Galois cohomology seminar
Week 6 - Galois cohomology, absolute Galois group, Hilbert 90

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Note on sources

The main sources for these notes are sections 2.3-2.4 of Sharifi [4], chapter 1 appendix A and chapter 2 (especially 1.22, 1.23, 1.24) of Milne [2], and chapter 4 of Gille & Szamuely [1]. The material on Brauer groups is not intended to be a proper introduction to central simple algebras. More details can be found in Rapinchuk [3].

Contents

1 Galois correspondence for infinite extensions

Remark 1.1. Let L/K be a Galois extension, and $\mathcal{E}$ be the set of intermediate subfields $K \subset E \subset L$ such that E/K is finite Galois. Then

$$L = \bigcup_{E \in \mathcal{E}} E$$

Additionally, $\mathcal{E}$ is partially ordered by inclusion, and is a directed set, since the compositum EE'/K is a finite Galois extension containing E and E'. If $E \subset E'$, then we have a restriction map $\text{Gal}(E'/K) \to \text{Gal}(E/K)$ by restricting automorphisms of E' to E. This makes the Galois groups $\text{Gal}(E/K)$ into a directed system.

Proposition 1.1. *Let L/K be a Galois extension. Then*

$$\mathrm{Gal}(L/K) \to \varprojlim \mathrm{Gal}(E/K) \qquad \sigma \mapsto (\sigma|_E)$$

where E ranges over intermediate subfields $K \subset E \subset L$ such that E/K is finite Galois.

Proof. Exercise for the reader to check that this actually maps into the direct limit, because I feel lazy right now. This is clearly a group homomorphism. It is also clear that the kernel is trivial, since if σ restricts to the identity on each E, it is the identity on L, since $L = \bigcup E$.

All that remains is surjectivity. Consider $(\sigma_E) \in \varprojlim \mathrm{Gal}(E/K)$. Define $\sigma : L \to L$ by $\sigma(x) = \sigma_E(x)$ for $x \in E$. By the compatibility condition of (σ_E) being in the inverse limit, if x lies in two fields E, E' then $\sigma_E(x) = \sigma_{E'}(x) = \sigma_{EE'}(x)$ so this is well defined. Since $L = \bigcup E$, this defines σ on all of L. Clearly σ restricts to σ_E for each E, so (σ_E) is in the image. □

Remark 1.2. Since the inverse limit has a natural topology as a profinite group, the isomorphism above makes $\mathrm{Gal}(L/K)$ a topological group. In the case where L/K is finite, this is just the discrete topology, but when L/K is infinite, this gives it a nontrivial topology. Whenever L/K is Galois, we assume that $\mathrm{Gal}(L/K)$ has this topology, called the **Krull topology**.

Theorem 1.2 (Fundamental theorem of Galois theory)**.** *Let L/K be a Galois extension and let $G = \mathrm{Gal}(L/K)$. There is a bijection*

$$\{\text{closed subgroups } H \subset G\} \longleftrightarrow \{\text{intermediate subfields } K \subset E \subset L\}$$
$$H \mapsto L^H$$
$$\mathrm{Gal}(L/E) \leftarrow\!\shortmid E$$

In particular, $L^{\mathrm{Gal}(L/E)} = E$ and $\mathrm{Gal}(L/L^H) = H$. Additionally,

1. *The correspondence is inclusion reversing, i.e. $H_1 \subset H_2 \iff L^{H_1} \supset L^{H_2}$.*
2. *A closed subgroup $H \subset G$ is normal if and only if L^H/K is Galois. In this case,*

$$\mathrm{Gal}(L^H/K) \cong \mathrm{Gal}(L/K)/H$$

3. *A closed subgroup $H \subset G$ is open if and only if L^H/K is a finite extension. In this case,*

$$[G : H] = [L^H : K]$$

Proof. Assuming the correspondence for finite Galois theory, this isn't too much work to show. This is not the focus of our seminar, so I'll skip over it. □

1.1 Absolute galois groups

Definition 1.1. Let K be a field. A **separable closure** of K is a field K^{sep} which contains all roots of separable polynomials over K.

Remark 1.3. In characteristic zero or for finite fields, separable closure is equal to algebraic closure. The separable closure exists and is unique up to isomorphism. Note that K^{sep}/K is Galois.

Definition 1.2. The **absolute Galois group** of K is $G_K = \text{Gal}(K^{\text{sep}}/K)$.

Proposition 1.3. *Let $K = \mathbb{F}_q$ be a finite field with q elements. Then $G_K \cong \widehat{\mathbb{Z}}$.*

Proof. For each $n \geq 1$, there is a unique finite extension of $\mathbb{F}_q$ of degree n, which is $\mathbb{F}_{q^n}$. Furthermore, the Galois group is

$$\text{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{Z}/n\mathbb{Z}$$

The inclusion relation on $\mathbb{F}_{q^n}$ is by divisibility of n, so the inverse system of Galois groups are the groups $\mathbb{Z}/n\mathbb{Z}$ for $n \geq 1$ ordered by divisibility with quotient maps $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ when $m|n$. This inverse limit of this, as we already know, is $\widehat{\mathbb{Z}}$. □

2 Galois cohomology

Definition 2.1. Let L/K be a Galois extension, and let A be a discrete $\text{Gal}(L/K)$-module (with respect to Krull topology). The i**th Galois cohomology group** is $H^i(\text{Gal}(L/K), A)$.

Remark 2.1. The most obvious modules for $\text{Gal}(L/K)$ are $(L, +)$ and $(L^\times, \times)$. We claim these are discrete modules. To show this, it suffices to show that the stabilizer of a point in L is open in $\text{Gal}(L/K)$.

$$\text{stab}(\alpha) = \{\sigma \in \text{Gal}(L/K) \mid \sigma\alpha = \alpha\} = \text{Gal}(L/K(\alpha))$$

By the Galois correspondence 1.2 item (3), this is open since $K(\alpha)/K$ is finite.

Recall a result that Stan proved last week.

Proposition 2.1. *Let G be a profinite group, and let $\mathcal{U}$ be the set of open normal subgroups of G. Let A be a discrete G-module. Then*

$$H^i(G, A) \cong \varinjlim_{N \in \mathcal{U}} H^i(G/N, A^N)$$

where the maps of the directed system are inflation maps.

Remark 2.2. In particular, in the case where L/K is a Galois extension and $G = \text{Gal}(L/K)$, by the Galois correspondence, the set of open normal subgroups of G is the set of subgroups $\text{Gal}(L/E)$ where E/K is finite Galois. For the modules $(L, +)$ and $(L^\times, \times)$, the above isomorphism is

$$H^i(\text{Gal}(L/K), L^\times) \cong \varinjlim_{E \in \mathcal{E}} H^i\left(\frac{\text{Gal}(L/K)}{\text{Gal}(L/E)}, (L^\times)^{\text{Gal}(L/E)}\right) \cong \varinjlim_{E \in \mathcal{E}} H^i(\text{Gal}(E/K), E^\times)$$

$$H^i(\text{Gal}(L/K), L) \cong \varinjlim_{E \in \mathcal{E}} H^i\left(\frac{\text{Gal}(L/K)}{\text{Gal}(L/E)}, L^{\text{Gal}(L/E)}\right) \cong \varinjlim_{E \in \mathcal{E}} H^i(\text{Gal}(E/K), E)$$

2.1 Multiplicative Hilbert 90

We recall a result from graduate algebra.

Lemma 2.2 (Linear independence of characters)**.** *Let M be a monoid (or group) and L be a field. Let $\chi_1, \ldots, \chi_n : M \to L$ be distinct monoid homomorphisms (where L is a monoid with respect to multiplication). Then $\chi_1, \ldots, \chi_n$ are linearly independent. That is, if we have a linear combination*

$$\sum_i a_i \chi_i \qquad a_i \in L$$

which is the zero morphism, then $a_i = 0$ for each i.

Theorem 2.3 (Generalization of multiplicative Hilbert 90)**.** *Let L/K be a Galois extension. Then*

$$H^1(\mathrm{Gal}(L/K), L^\times) = 0$$

Proof. By Remark 2.2,

$$H^1(\mathrm{Gal}(L/K), L^\times) \cong \varinjlim H^1(\mathrm{Gal}(E/K), E^\times)$$

where E ranges over finite Galois subextensions. Thus if we prove the result for finite extensions, it follows since the direct limit of trivial groups is trivial. So we may assume L/K is finite (so G is finite).

For clarity, we write $\cdot$ for multiplication in $L^\times$. Let $f : G \to L^\times$ be a cocycle, that is, for $\tau, \sigma \in G$, [1]

$$f(\tau\sigma) = \tau\Big(f(\sigma)\Big) \cdot f(\tau)$$

The elements $\sigma \in G$ are distinct characters $L^\times \to L$, so they are linearly independent by Lemma 2.2. Thus

$$\sum_{\sigma \in G} f(\sigma)\sigma$$

is a nonzero map (since $f(\sigma) \neq 0$). Let $\alpha \in L^\times$ so that

$$\beta = \sum_{\sigma \in G} f(\sigma) \cdot \sigma(\alpha) \neq 0$$

[1]This may be confusing, since usually the cocycle condition would be written $f(\tau\sigma) = \tau f(\sigma) + f(\tau)$ but thisis when the G-module is written additively, and here we are writing our G-module $L^\times$ multiplicatively.

Then for $\tau \in G$,

$$\begin{aligned}
\tau^{-1}(\beta) &= \tau^{-1} \sum_{\sigma \in G} f(\sigma) \cdot \sigma(\alpha) \\
&= \sum_{\sigma \in G} \tau^{-1}\Big(f(\sigma) \cdot \sigma(\alpha)\Big) && \text{linearity} \\
&= \sum_{\sigma \in G} \Big(\tau^{-1} f(\sigma)\Big) \cdot \Big(\tau^{-1}\sigma(\alpha)\Big) && \tau \text{ is a field hom} \\
&= \sum_{g \in G} \Big(\tau^{-1} f(\tau g)\Big) \cdot g(\alpha) && \text{substitute } g = \tau^{-1}\sigma \\
&= \sum_{\sigma \in G} \Big(\tau^{-1} f(\sigma)\Big) \cdot \sigma(\alpha) && \text{substitute } g = \sigma \\
&= \sum_{\sigma \in G} \tau^{-1}\Big(f(\tau) \cdot \tau\big(f(\sigma)\big)\Big) \cdot \sigma(\alpha) && f \text{ is a cocycle} \\
&= \sum_{\sigma \in G} \tau^{-1}(f(\tau)) \cdot f(\sigma) \cdot \sigma(\alpha) && \tau \text{ is a field hom} \\
&= \tau^{-1}\Big(f(\tau)\Big) \cdot \sum_{\sigma \in G} f(\sigma) \cdot \sigma(\alpha) && \text{linearity} \\
&= \tau^{-1}\Big(f(\tau)\Big) \cdot \beta
\end{aligned}$$

Applying τ to both sides,

$$\beta = f(\tau) \cdot \tau(\beta) \qquad f(\tau) = \frac{\beta}{\tau(\beta)} = \frac{\tau(\beta^{-1})}{\beta^{-1}}$$

Thus f is a coboundary. [2] □

Theorem 2.4 (Classical multiplicative Hilbert 90)**.** *Let L/K be a finite cyclic Galois extension, let $N_K^L : L^\times \to K^\times$ be the norm map, and let $\sigma \in \operatorname{Gal}(L/K)$ be a generator. Then*

$$\ker N_K^L = \left\{ \frac{\sigma\beta}{\beta} \;\middle|\; \beta \in L^\times \right\}$$

Proof. Let $G = \operatorname{Gal}(L/K)$ and $n = [L : K] = |G|$. Since σ generates G, the element $\sigma - 1 \in \mathbb{Z}[G]$ generates I_G, hence right hand side is exactly $I_G L^\times$. Thus the claim is equivalent to either of the following.

$$\ker N_K^L = I_G L^\times \qquad \ker N_K^L / I_G L^\times = 0$$

Note that the field norm map N_K^L coincides with the group norm map N_G, as shown below.

$$N_G(\beta) = \left(\sum_{i=0}^{n-1} \sigma^i \right) \beta = \prod_{i=0}^{n-1} (\sigma^i \beta) = N_K^L(\beta)$$

[2] Again, this looks a bit strange since things are written multiplicatively instead of additively, but it is right. The usual coboundary condition for f to be a degree one coboundary is that there exists x in the module such that $f(\tau) = \tau(x) - x$, but in multiplicative notation it becomes $f(\tau) = \frac{\tau x}{x}$.

(Since we write $L^\times$ multiplicatively, the sum becomes a product). By definition of Tate cohomology,

$$\widehat{H}^{-1}(G, L^\times) = \ker N_G / I_G L^\times = \ker N_K^L / I_G L^\times$$

Thus the claim reduces to showing $\widehat{H}^{-1}(G, L^\times) = 0$. Since G is cyclic,

$$\widehat{H}^{-1}(G, L^\times) \cong \widehat{H}^1(G, L^\times) = H^1(G, L^\times) = 0$$

with the final equality from Theorem 2.3. □

2.2 Additive Hilbert 90

Theorem 2.5 (Normal basis theorem)**.** *Let L/K be a finite Galois extension. Then there exists $\alpha \in L$ such that*

$$\{\sigma(\alpha) : \sigma \in \mathrm{Gal}(L/K)\}$$

is a K-basis of L.

Proof. (Not a proof.) Usually the proof is broken into cases where K is finite/infinite. The finite case is not hard, since in that case G is cyclic. □

Remark 2.3. Let L/K be a finite Galois extension and $G = \mathrm{Gal}(L/K)$. Then L is a $K[G]$-module via

$$K[G] \times L \to L \qquad \left(\sum_{\sigma \in G} \lambda_\sigma \sigma\right) \cdot x = \sum_{\sigma \in G} \lambda_\sigma \sigma(x)$$

where $\lambda_\sigma \in K$. Another way to interpret the normal basis theorem is that $L \cong K[G]$ as a $K[G]$-module. Let α be the element of the normal basis theorem. Then

$$L \to K[G] \qquad \sum_{\sigma \in G} \lambda_\sigma \sigma(\alpha) \mapsto \sum_{\sigma \in G} \lambda_\sigma \sigma$$

is an isomorphism of $K[G]$-modules.

Theorem 2.6 (Generalized additive Hilbert 90)**.** *Let L/K be a Galois extension. Then*

$$H^i(\mathrm{Gal}(L/K), L) = 0$$

for all $i \geq 1$.

Proof. As in the previous proof, Remark 2.2, which says

$$H^i(\mathrm{Gal}(L/K), L) \cong \varinjlim H^i(\mathrm{Gal}(E/K), E)$$

allows us to reduce to the case of a finite Galois extension. So assume L/K is finite, and let $G = \mathrm{Gal}(L/K)$. By the normal basis theorem, $L \cong K[G]$ as a $K[G]$-module, and this is also an isomorphism of G-modules. Thus

$$L \cong K[G] \cong \mathbb{Z}[G] \otimes_{\mathbb{Z}} K \cong \mathrm{Ind}^G(K) \qquad (\cong \text{ of G-modules})$$

Thus $H^i(G, L) = 0$ for $i \geq 1$ since cohomology always vanishes for induced/coinduced modules. □

3 Brauer groups

Definition 3.1. Let K be a field, and fix a separable closure K^{sep}. The **Brauer group** of K is

$$\text{Br}(K) = H^2(G_K, (K^{\text{sep}})^\times) = H^2\left(\text{Gal}(K^{\text{sep}}/K), (K^{\text{sep}})^\times\right)$$

Remark 3.1. You may be familiar with another description of $\text{Br}(K)$ in terms of central simple algebras. These are isomorphic, but that takes a bit of work to show. Here is a whirlwind tour of that alternate description of $\text{Br}(K)$.

- A central K-algebra is one whose center is exactly K.
- A simple K-algebra is one with no nontrivial two sided ideals.
- Elements of $\text{Br}(K)$ are given by isomorphism classes of central simple K-algebras.
- The group operation in $\text{Br}(K)$ is given by tensor product over K.

$$[A] \cdot [B] = [A \otimes_K B]$$

- The identity is represented by the algebra K (actually by $M_n(K)$ for any n).
- The inverse of $[A]$ is represented by the opposite algebra A^{op}.

This mostly concludes our discussion $\text{Br}(K)$ in terms of central simple algebras.

Recall the inflation-restriction sequence result for regular (non-profinite cohomology).

Proposition 3.1. *Let G be a group and $N \subset G$ a normal subgroup and A a G-module. Let $i \geq 1$, and suppose that $H^j(N, A) = 0$ for all $1 \leq j \leq i - 1$. Then the following sequence is exact.*

$$0 \longrightarrow H^i(G/N, A^N) \xrightarrow{\text{Inf}} H^i(G, A) \xrightarrow{\text{Res}} H^i(N, A)$$

Remark 3.2. If G is a profinite group, and $N \subset G$ is a closed normal subgroup, and A is a discrete G-module, then the preceeding result holds for profinite cohomology. This follows from the regular cohomology result using the isomorphism

$$H^i(G, A) \cong \varinjlim H^i(G/H, A^H)$$

along with exactness of the direct limit functor. I suppose this depends on how we define inflation and restriction for profinite cohomology. If we want this to be really obvious, then we can define them to be the direct limit of inflation and restriction maps on cohomology of finite groups.

Proposition 3.2. *Let G be a **profinite** group and $N \subset G$ a **closed** normal subgroup and A a **discrete** G-module. Let $i \geq 1$, and suppose that $H^j(N, A) = 0$ for all $1 \leq j \leq i - 1$. Then the following sequence is exact.*

$$0 \longrightarrow H^i(G/N, A^N) \xrightarrow{\text{Inf}} H^i(G, A) \xrightarrow{\text{Res}} H^i(N, A)$$

Proposition 3.3. *Let L/K be a Galois extension. There is an exact sequence of abelian groups*

$$0 \longrightarrow H^2(\mathrm{Gal}(L/K), L^\times) \xrightarrow{\mathrm{Inf}} \mathrm{Br}(K) \xrightarrow{\mathrm{Res}} \mathrm{Br}(L)$$

Proof. Fix a separable closure L^{sep} of L.

$$\begin{array}{c} L^{\mathrm{sep}} = K^{\mathrm{sep}} \\ | \\ L \\ | \\ K \end{array}$$

By the multiplicative version of Hilbert 90, $H^1(\mathrm{Gal}(K^{\mathrm{sep}}/K), (K^{\mathrm{sep}})^\times) = 0$, so we have the Inf / Res exact sequence in the case $i = 2, A = (K^{\mathrm{sep}})^\times = (L^{\mathrm{sep}})^\times, G = G_K = \mathrm{Gal}(K^{\mathrm{sep}}/K), N = G_L = \mathrm{Gal}(L^{\mathrm{sep}}/L)$.

$$\begin{array}{ccccccc} 0 \longrightarrow & H^2(G_K/G_L, ((K^{\mathrm{sep}})^\times)^{G_L}) & \xrightarrow{\mathrm{Inf}} & H^2(G_K, (K^{\mathrm{sep}})^\times) & \xrightarrow{\mathrm{Res}} & H^2(G_L, (K^{\mathrm{sep}})^\times) \\ & = & & = & & = \\ & H^2(\mathrm{Gal}(L/K), L^\times) & & \mathrm{Br}(K) & & H^2(G_L, (L^{\mathrm{sep}})^\times) \\ & & & & & = \\ & & & & & \mathrm{Br}(L) \end{array}$$

The simplification of the first term utilizes the following isomorphisms of the Galois correspondence.

$$\begin{aligned} \Big((K^{\mathrm{sep}})^\times\Big)^{G_L} &= \Big((K^{\mathrm{sep}})^\times\Big)^{\mathrm{Gal}(K^{\mathrm{sep}}/L)} = L^\times \\ \frac{G_K}{G_L} &= \frac{\mathrm{Gal}(K^{\mathrm{sep}}/K)}{\mathrm{Gal}(L^{\mathrm{sep}}/L)} = \frac{\mathrm{Gal}(K^{\mathrm{sep}}/K)}{\mathrm{Gal}(K^{\mathrm{sep}}/L)} \cong \mathrm{Gal}(L/K) \end{aligned}$$

□

Remark 3.3. In the language of central simple algebras, the map $\mathrm{Res} : \mathrm{Br}(K) \to \mathrm{Br}(L)$ has a simple description, which is just

$$\mathrm{Br}(K) \to \mathrm{Br}(L) \qquad A \mapsto A \otimes_K L$$

The kernel of this, which we now know coincides with $H^2(\mathrm{Gal}(L/K), L^\times)$, is called the **relative Brauer group** $\mathrm{Br}(L/K)$, so we can rewrite the above exact sequence as

$$0 \to \mathrm{Br}(L/K) \to \mathrm{Br}(K) \to \mathrm{Br}(L)$$

3.1 Brauer group of a finite field

Example 3.1. We show that the Brauer group of a finite field is trivial. Let q be a prime power, and let $\mathbb{F}_q$ be the field with q elements. By our result about direct limits, we just need to show that H^2 is trivial for the finite Galois subextensions. As we noted earlier, the finite Galois extensions of $\mathbb{F}_q$ are $\mathbb{F}_{q^n}$ for $n \geq 1$, with cyclic Galois groups. Let

$$G_n = \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \mathbb{Z}/n\mathbb{Z}$$

Then we have

$$\mathrm{Br}(\mathbb{F}_q) = \mathrm{Gal}(\mathbb{F}_q^{\mathrm{sep}}/\mathbb{F}_q), \mathbb{F}_{q^n}^\times) = \varinjlim H^2(\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q), \mathbb{F}_{q^n}^\times) = \varinjlim H^2(G_n, \mathbb{F}_{q^n}^\times)$$

Let $N = N_{G_n}$ be the norm element, and recall that multiplication by the norm element of G_n is the same as the field norm map

$$N_G = N_{\mathbb{F}_q}^{\mathbb{F}_{q^n}} : \mathbb{F}_{q^n}^\times \to \mathbb{F}_q^\times$$

We know that the Tate cohomology is 2-periodic for finite cyclic groups. Using this and the fact that the fixed field of G_n is exactly $\mathbb{F}_q$, we get

$$H^2(G_n, \mathbb{F}_{q^n}^\times) \cong \widehat{H}^0(G_n, \mathbb{F}_{q^n}^\times) \cong \left(\mathbb{F}_{q^n}^\times\right)^{G_n} / N_G \mathbb{F}_{q^n}^\times = \mathbb{F}_q^\times / \operatorname{im} N_{\mathbb{F}_q}^{\mathbb{F}_{q^n}}$$

Thus, we have reduced the problem to showing that the norm map is surjective for finite fields. If we show this, then we have shown $\mathrm{Br}(\mathbb{F}_q) = 0$.

Lemma 3.4. *The norm map for finite fields is surjective.*

$$N_{\mathbb{F}_q}^{\mathbb{F}_{q^n}} : \mathbb{F}_{q^n}^\times \twoheadrightarrow \mathbb{F}_q^\times$$

Proof. Recall that $\mathbb{F}_q^\times$ consists of qth roots of unity. Similarly, $\mathbb{F}_{q^n}$ consists of $(q^n - 1)$th roots of unity. Recall that the Galois group $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ is generated by the Frobenius automorphism

$$\phi : \mathbb{F}_{q^n} \to \mathbb{F}_{q^n} \qquad x \mapsto x^q$$

Let $\alpha \in \mathbb{F}_{q^n}^\times$ be a primitive $(q^n - 1)$th root of unity, that is, a generator of $\mathbb{F}_{q^n}^\times$. The norm is the product of the Galois conjugates, so

$$N_{\mathbb{F}_q}^{\mathbb{F}_{q^n}}(\alpha) = \prod_{\sigma \in G} \sigma(\alpha) = \prod_{i=0}^{n-1} \phi^i(\alpha) = \prod_{i=0}^{n-1} \alpha^{q^i} = \alpha^{1+q+q^2+\cdots+q^{n-1}} = \alpha^{\frac{q^n-1}{q-1}}$$

The last equality comes from the formula for the sum of a finite geometric series. Then observe that

$$\left(\alpha^{\frac{q^n-1}{q-1}}\right)^{q-1} = \alpha^{q^n-1} = 1$$

by definition of α. That is, the image of α under the norm map is a primitive $(q-1)$th root of unity, so it is a generator of $\mathbb{F}_q^\times$. Thus the nrom map is surjective. $\square$

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